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On Lie's Theory of Continuous Groups.

BY E. W. RETTGER.

An r-parameter group generated by the infinitesimal transformations $X_1, X_2, \ldots, X_r$ is continuous if each transformation of the group, without exception, belongs to a one-parameter group whose equations, at least in the neighborhood of the identical transformations, are

$$x_i' = x_i + \sum_1^r {}^j \, l_j X_j x_i + \frac{1}{1 \cdot 2} \sum_1^r {}^j \sum_1^r {}^k l_j l_k X_j X_k x_i + \ldots, \qquad (1)$$
$$(i = 1, 2, \ldots, n)$$

that is to say, if each transformation of the group can be generated by an infinitesimal transformation of the group.

In his "Continuerliche Gruppen," Lie demonstrated that every transformation of the general projective group can be generated by an infinitesimal transformation of this group. Whence it follows that this group is continuous.* But Professor Study's important discovery † that not every transformation of the special linear homogeneous group can be generated by an infinitesimal transformation of the special linear homogeneous group, shows that not every subgroup of the general projective group is continuous, as was apparently assumed by Lie.

In this paper I shall term a transformation of an r-parameter group G_r that cannot be generated by the repetition of an infinitesimal transformation of this group a *singular transformation* of the group G_r. Study pointed out that the special linear homogeneous group in two variables contained singular transformations. Subsequently Professor Taber established the existence of singular transformations in the group of orthogonal substitutions in n variables, for $n \geqq 4$, and also in the group of linear automorphic transformations of an alternate bilinear form, and the group of linear automorphic transformations of a general bilinear form.*

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* Lie, "Continuerliche Gruppen," p. 45.

† "Leipzige Berichte," 1892.

In what follows I shall investigate the two- and three-parameter subgroups of the general projective group in two variables, and of the general linear homogeneous group in three variables, enumerated by Lie on pages 288 and 519 respectively, of his "Continuerliche Gruppen;" and I propose to show that singular transformations occur among the transformations of many of these subgroups.

Let $X_1, X_2, \ldots, X_r$ be the r infinitesimal transformations that generate a given r-parameter group G_r. I shall denote by T_a the transformation defined by the equations

$$x_i' = x_i + \sum_1^r{}^j a_j X_j x_i + \frac{1}{1 \, . \, 2} \sum_1^r{}^j \sum_1^r{}^k a_j a_k X_j X_k x_i + \ldots \ldots,$$
$$(i = 1, 2, \ldots \ldots, n)$$

and, therefore, generated by the infinitesimal transformation whose symbol is

$$a_1 X_1 + \ldots \ldots + a_r X_r.$$

Similarly, T_b and T_c will denote the transformations defined respectively by

$$x_i' = x_i + \sum_1^r{}^j b_j X_j x_i \frac{1}{1 \, . \, 2} \sum_1^r{}^j \sum_1^r{}^k b_j b_k X_j X_k x_i + \ldots \ldots,$$
$$(i = 1, 2, \ldots \ldots, n)$$

and

$$x_i' = x_i + \sum_1^r{}^j c_j X_j x_i + \frac{1}{1 \, . \, 2} \sum_1^r{}^j \sum_1^r{}^k c_j c_k X_j X_k x_i + \ldots \ldots,$$
$$(i = 1, 2, \ldots \ldots, n)$$

and, therefore, generated respectively by the infinitesimal transformations

$$b_1 X_1 + \ldots \ldots + b_r X_r,$$

and

$$c_1 X_1 + \ldots \ldots + c_r X_r.$$

Furthermore, $T_b T_a$ will denote the composition of T_a and T_b in the order named, that is, the transformation resulting from the successive application to the manifold $(x_1, x_2, \ldots., x_n)$ of the transformations T_a and T_b in the order named.*

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* Am. Journ. Math., vol. XVI, p. 130; Bul. N. Y. Math. Soc., July, 1894; Math. Ann., vol. XLVI, p. 561; Math. Review, vol. I, p. 154.

In general, whatever be the values of the a's and b's, we can put

$$T_b\,T_a = T_c,$$

where
$$c_k = \lambda_k(a_1 \ldots. a_r\ b_1 \ldots. b_r); \qquad (2)$$
$$(k = 1, 2, \ldots., r)$$

that is to say, the composition of two transformations T_a and T_b generated respectively by the infinitesimal transformations $aX_1 + \ldots. + a_r X_r$, and $b_1 X_1 + \ldots. + b_r X_r$, results, in general, in a transformation, T_c, which can also be generated by an infinitesimal transformation, viz.:

$$c_1 X_1 + \ldots. + c_r X_r.$$

Now, a transformation of G_r can often be generated by more than one infinitesimal transformation of G_r. In this case, equations (2) give more than one set of values for the c's that correspond to the particular set of values assigned to the a's and b's; that is to say, *the c's are multivalued functions of the a's and b's.* Let $T_b\,T_a$ be finite for $a_j = \bar{a}_j$ and $b_j = \bar{b}_j$, $(j = 1, 2 \ldots. r)$. It may be, for this system of values of the a's and b's, that—

1st. Every branch of each function $\lambda_1 \ldots. \lambda_r$ is finite and determinate;

2nd. One or more, but not all, of the branches of one (or more) of the functions $\lambda_1 \ldots. \lambda_r$ is infinite or indeterminate;

3rd. Each branch of one (or more) of the functions $\lambda_1 \ldots. \lambda_r$ is infinite or indeterminate.

Only in the last case do equations (2) fail to give us the parameters of an infinitesimal transformation which will generate $T_b\,T_a$. In this case, there is no infinitesimal transformation of G_r that will generate $T_b\,T_a$; that is to say, $T_b\,T_a$ is a *singular transformation.*

To ascertain whether G_r contains singular transformations, we have then (knowing $\lambda_1 \ldots. \lambda_r$) to ascertain whether one (or more) sets of values of the a's and b's will make each branch of one (or more) of the functions $\lambda_1 \ldots. \lambda_r$

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*Lie denotes by $T_b\,T_a$, the transformation resulting from the application to the manifold $(x_1, x_2, \ldots., x_n)$, first of the transformation T_b, and then of the transformation T_a.

infinite or indeterminate. If no such system of values of the a's and b's can be found, G_r contains no singular transformation. If, on the other hand, one or more such systems of values of the a's and b's can be found, then the group G_r contains one or more singular transformations.

To determine $\lambda_1, \lambda_2, \ldots\ldots, \lambda_r$, let the summation of the infinite series

$$x_i' = x_i + \sum_1^r{}^j\, l_j X_j x_i + \frac{1}{1\,.\,2}\sum_1^r{}^j \sum_1^r{}^k\, l_j l_k X_j X_k x_i + \ldots\ldots \tag{1}$$

$$(i = 1,\ 2,\ \ldots\ldots,\ n)$$

give
$$x_i' = \phi_i(x_1 \ldots\ldots x_n,\ l_1 \ldots\ldots l_r); \qquad (i = 1,\ 2,\ \ldots\ldots,\ n)$$

or, in the abbreviated notation,
$$x_i' = \phi_i(x,\ l). \qquad (i = 1,\ 2,\ \ldots\ldots,\ n) \tag{3}$$

Then T_a and T_b are defined respectively by the equations

$$\left.\begin{aligned} x_i' &= \phi_i(x,\ a) \\ \text{and}\qquad x_i'' &= \phi_i(x',\ b). \end{aligned}\right\} \qquad (i = 1,\ 2,\ \ldots\ldots,\ n)$$

Eliminating $x_1' \ldots\ldots x_n'$ from the last two systems of equations, we obtain, as a consequence of the fundamental property of a group,

$$x_i'' = \phi_i\big(x,\ \lambda(ab)\big). \qquad (i = 1,\ 2,\ \ldots\ldots,\ n)$$

If the transformation defined by this system of equations is identical with the transformation T_c, this system of equations is equivalent to the system

$$x_i'' = \phi_i(x,\ c); \qquad (i = 1,\ 2,\ \ldots\ldots,\ n)$$

and, from this identity, we derive the system of equations

$$c_k = \lambda_k(a,\ b). \qquad (k = 1,\ 2,\ \ldots\ldots,\ r) \tag{2}$$

I shall apply the foregoing to several subgroups of the general projective group of the plane.

The subgroup, $q,\ yq,\ xp$.* For this group $\sum_1^r{}^j\, l_j X_j \equiv l_1 q + l_2 yq + l_3 xp$, and equations (1) become

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* Lie employs p and q to denote respectively $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$x' = x + l_3 x + \tfrac{1}{2} l_3^2 x + \ldots\ldots = e^{l_3} x,$$
$$y' = y + l_1 + l_2 y + \frac{l_2 l_1 + l_2^2 y}{2} + \ldots\ldots = e^{l_2} y + \frac{l_1}{l_2}(e^{l_2} - 1).$$

Consequently, T_a and T_b are defined respectively by the equations

$$\left.\begin{aligned} x' &= e^{a_3} x, \\ y' &= e^{a_2} y + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}\right\} \qquad \text{(A)}$$

and

$$\left.\begin{aligned} x'' &= e^{b_3} x', \\ y'' &= e^{b_2} y' + \frac{b_1}{b_2}(e^{b_2} - 1). \end{aligned}\right\} \qquad \text{(B)}$$

Eliminating x' and y' between equations (A) and (B), we obtain, as the system of equations defining $T_b\,T_a$,

$$\left.\begin{aligned} x'' &= e^{a_3 + b_3} x, \\ y'' &= e^{a_2 + b_2} y + N, \end{aligned}\right\} \qquad \text{(C)}$$

where N denotes

$$e^{b_2} \frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1).$$

Moreover, T_c is defined by the system of equations

$$\left.\begin{aligned} x'' &= e^{c_3} x, \\ y'' &= e^{c_2} y + \frac{c_1}{c_2}(e^{c_2} - 1). \end{aligned}\right\} \qquad \text{(D)}$$

Therefore, if $T_b\,T_a = T_c$, that is, if the system of equations (C) is equivalent to the system of equations (D),

$$e^{c_3} = e^{a_3 + b_3},$$
$$e^{c_2} = e^{a_2 + b_2},$$
$$\frac{c_1}{c_2}(e^{c_2} - 1) = N.$$

Consequently, equations (2) become

$$\left.\begin{aligned} c_3 &= a_3 + b_3 + 2m\pi i, \\ c_2 &= a_2 + b_2 + 2n\pi i, \\ c_1 &= \frac{N(a_2 + b_2 + 2n\pi i)}{e^{a_2 + b_2} - 1}, \end{aligned}\right\} \qquad \text{(E)}$$

where m and n are arbitrary integers or are equal to zero.

From equations (E), it follows that c_3 and c_2 are finite for all sets of finite values of the a's and b's. Moreover, c_1 is finite for finite values of the a's and b's, unless $a_2 + b_2$ is an even multiple, $2n'$, of πi. But in this case, if we put $n = -n'$, we have $c_1 = N$ which is certainly finite. Therefore, c_1, c_2, and c_3 can all be taken finite for finite values of the a's and b's. Whence, it follows that this group does not contain singular transformations. We have here an illustration of the 2nd case mentioned on page 62.

The subgroup, q, $yq+p$. For this group $\sum_1^r l_j X_j \equiv l_1 q + l_2(yq+p)$; and equations (1) become

$$x' = x + l_2,$$
$$y' = e^{l_2} y + \frac{l_1}{l_2}(e^{l_2} - 1).$$

Consequently, T_a and T_b are defined respectively by

$$\left.\begin{aligned} x' &= x + a_2, \\ y' &= e^{a_2} y + \frac{a_1}{a_2}(e^{a_2} - 1), \end{aligned}\right\} \qquad \text{(A)}$$

and

$$\left.\begin{aligned} x'' &= x' + b_2, \\ y'' &= e^{b_2} y' + \frac{b_1}{b_2}(e^{b_2} - 1); \end{aligned}\right\} \qquad \text{(B)}$$

and $T_b\, T_a$ by

$$\left.\begin{aligned} x'' &= x + a_2 + b_2, \\ y'' &= e^{a_2 + b_2} y + N, \end{aligned}\right\} \qquad \text{(C)}$$

where N denotes $e^{b_2} \frac{a_1}{a_2}(e^{a_2} - 1) + \frac{b_1}{b_2}(e^{b_2} - 1)$. Moreover, T_c is defined by

$$\left.\begin{aligned} x'' &= x + c_2, \\ y'' &= e^{c_2} y + \frac{c_1}{c_2}(e^{c_2} - 1). \end{aligned}\right\} \qquad \text{(D)}$$

Therefore, if $T_b\, T_a = T_c$,

$$c_2 = a_2 + b_2,$$
$$\frac{c_1}{c_2}(e^{c_2} - 1) = N.$$

Consequently, equations (2) become

$$\left.\begin{aligned} c_2 &= a_2 + b_2, \\ c_1 &= \frac{N(a_2 + b_2)}{e^{a_2 + b_2} - 1}. \end{aligned}\right\} \tag{E}$$

From the first of equations (E), it follows that c_2 is finite for all finite values of the a's and b's. If $a_2 + b_2 = 0$, it follows from the last of these equations that c_1, which in this case equals N, is finite; but if $a_2 + b_2 = 2m\pi i$ for some integer $m \neq 0$, c_1 is infinite, provided $N \neq 0$. Now, we can so choose the a's and b's that $a_2 + b_2 = 2m\pi i \neq 0$, and at the same time that $N \neq 0$. For let $a_2 + b_2 = 2m\pi i \neq 0$, then

$$N = \left(\frac{a_1}{a_2} - \frac{b_1}{b_2}\right)(1 - e^{-a_2}).$$

By properly choosing the a's and b's, subject to the negative condition that $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ and that a_2 is not an integer-multiple of $2\pi i$, N may be taken arbitrarily. We may, therefore, assume $N \neq 0$. We thus obtain a transformation $T_b\,T_a$ of our group,

$$\begin{aligned} x'' &= x + 2m\pi i \qquad && (m \text{ an integer } \neq 0) \\ y'' &= y + N, && (N \neq 0) \end{aligned}$$

which cannot be generated by an infinitesimal transformation of our group. Consequently, the given group contains singular transformations.

I shall now develop a few theorems, more or less evident, that will be made use of in this paper. Let the two transformations T_a and T_b of G_r, generated respectively by $\sum_1^r{}^s a_s X_s$ and $\sum_1^r{}^s b_s X_s$, be applied successively to the manifold $(x_1 \dots . x_n)$. We have then the simultaneous systems of equations

$$x_i' = x_i + \sum_1^r{}^s a_s X_s x_i + \tfrac{1}{2}\Big(\sum_1^r{}^s a_s X\Big)^2 x_i + \dots . \tag{4}$$

$$(i = 1, 2, \dots , n)$$

and

$$x_i'' = x_i' + \sum_1^r{}^s b_s X_s' x_i' + \tfrac{1}{2}\Big(\sum_1^r{}^s b_s X_s'\Big)^2 x_i' + \dots , \tag{5}$$

$$(i = 1, 2, \dots , n)$$

where X_s' denotes the result of substituting in X_s the x''s for the x's, and $\left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^2 x_i$, etc., denote respectively $\sum_{1}^{r}{}_{s}\, a_s X_s \left(\sum_{1}^{r}{}_{s}\, a_s X_s x_i\right)$, etc.

On eliminating $x_1', x_2', \ldots., x_n'$ from the systems of equations (4) and (5), we obtain the equations*

$$x_i'' = x_i + \left(\sum_{1}^{r}{}_{s}\, a_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s\right) x_i + \frac{1}{1\,.\,2}\left[\left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^2 + 2\sum_{1}^{r}{}_{s}\, b_s X_s \,.\, \sum_{1}^{r}{}_{s}\, a_s X_s + \left(\sum_{1}^{r}{}_{s}\, b_s X_s\right)^2\right] x_i + \ldots., \qquad (6)$$

$$(i = 1,\ 2,\ \ldots.,\ n)$$

which define the transformation $T_b\, T_a$.

Let now the infinitesimal transformations $\sum_{1}^{r}{}_{s}\, a_s X_s$ and $\sum_{1}^{r}{}_{s}\, b_s X_s$ be commutative; that is, let $\left(\sum_{1}^{r}{}_{s}\, a_s X_s,\ \sum_{1}^{r}{}_{s}\, b_s X_s\right) = 0$; or, what is the same thing, let $\sum_{1}^{r}{}_{s}\, a_s X_s \,.\, \sum_{1}^{r}{}_{s}\, b_s X_s = \sum_{1}^{r}{}_{s}\, b_s X_s \,.\, \sum_{1}^{r}{}_{s}\, a_s X_s$. Then $\left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^2 + 2\sum_{1}^{r}{}_{s}\, b_s X_s \,.\, \sum_{1}^{r}{}_{s}\, a_s X_s + \left(\sum_{1}^{r}{}_{s}\, b_s X_s\right)^2 = \left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^2 + \sum_{1}^{r}{}_{s}\, a_s X_s \,.\, \sum_{1}^{r}{}_{s}\, b_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s \,.\, \sum_{1}^{r}{}_{s}\, a_s X_s + \left(\sum_{1}^{r}{}_{s}\, b_s X_s\right)^2$ $= \left(\sum_{1}^{r}{}_{s}\, a_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s\right) . \left(\sum_{1}^{r}{}_{s}\, a_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s\right) = \left(\sum_{1}^{r}{}_{s}\, a_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s\right)^2$ $= \left(\sum_{1}^{r}{}_{s}\, \overline{a_s + b_s}\, X_s\right)^2$. Similarly, $\left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^3 + 3\left(\sum_{1}^{r}{}_{s}\, b_s X_s\right) . \left(\sum_{1}^{r}{}_{s}\, a_s X_s\right)^2$ $+ 3\left(\sum_{1}^{r}{}_{s}\, b_s X_s\right)^2 . \sum_{1}^{r}{}_{s}\, a_s X_s + \left(\sum_{1}^{r}{}_{s}\, b_s X_s\right)^3 = \left(\sum_{1}^{r}{}_{s}\, a_s X_s + \sum_{1}^{r}{}_{s}\, b_s X_s\right)^3 = \left(\sum_{1}^{r}{}_{s}\, \overline{a_s + b_s}\, X_s\right)^3$, etc. Consequently, if $\left(\sum_{1}^{r}{}_{s}\, a_s X_s,\ \sum_{1}^{r}{}_{s}\, b_s X_s\right) = 0$, equations (6) may be written

$$x_i'' = x_i + \sum_{1}^{r}{}_{s}\, \overline{a_s + b_s}\, X_s x_i + \frac{1}{1\,.\,2}\left(\sum_{1}^{r}{}_{s}\, \overline{a_s + b_s}\, X_s\right)^2 x_i + \frac{1}{1\,.\,2\,.\,3}\left(\sum_{1}^{r}{}_{s}\, \overline{a_s + b_s}\, X_s\right)^3 x_i + \ldots;$$

$$(i = 1,\ 2,\ \ldots.,\ n)$$

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* Lie, "Continuerliche Gruppen," p. 437.

and hence, in this case, $T_b T_a = T_a T_b$.* If, moreover, $T_b T_a = T_c$, where T_c is defined by

$$x_i'' = x_i + \sum_{s=1}^{r} c_s X_s x_i + \frac{1}{1 \, . \, 2}\Big(\sum_{s=1}^{r} c_s X_s\Big)^2 x_i + \frac{1}{1 \, . \, 2 \, . \, 3}\Big(\sum_{s=1}^{r} c_s X_s\Big)^3 x_i + \ldots . ;$$
$$(i = 1, 2, \ldots . , n)$$

we may put $c_k = a_k + b_k$ for $k = 1, 2, \ldots . , r$; and, therefore, the c's are finite for finite values of the a's and b's. We have thus the following theorem:

THEOREM I.—*If the two infinitesimal transformations* $\sum_{s=1}^{r} a_s X_s$ *and* $\sum_{s=1}^{r} b_s X_s$ *which generate respectively the finite transformations* T_a *and* T_b *of* G_r *are commutative, or what is the same thing, if* $\Big(\sum_{s=1}^{r} a_s X_s, \sum_{s=1}^{r} b_s X_s\Big) = 0$, *then the transformation* $T_b T_a$ *resulting from their composition is equivalent to a non-singular transformation* T_c *generated by the infinitesimal transformation* $\sum_{s=1}^{r} c_s X_s$, *where*

$$c_k = a_k + b_k. \qquad (k = 1, 2, \ldots . , n)$$

If, now, $(X_j, X_k) = 0$ for $j, k = 1, 2, \ldots . , r$, then for *every* system of values of the a's and b's $\Big(\sum_{s=1}^{r} a_s X_s, \sum_{s=2}^{r} b_s X_s\Big) = 0$, and, therefore, by theorem I, $T_b T_a$ (which is now equivalent to $T_a T_b$ for every system of values of the a's and b's) is non-singular. Conversely, if $T_b T_a = T_a T_b$ for *every* system of values of the a's and b's, then $(X_j, X_k) = 0$ for $j, k = 1, 2, \ldots . , r$; and, consequently, $T_b T_a$ is non-singular. Whence we derive—

THEOREM II.—*If the transformations of* G_r *are all commutative, or, what is the same thing, if* $(X_j, X_k) = 0$ *for* $j, k = 1, 2, \ldots . , r$, *no transformation of* G_r *is singular.*

As a particular case, we have, since the transformations of a one-parameter group G_1 are commutative, *a one-parameter group,* G_1, *contains no singular transformations.*†

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* Lie, "Continuerliche Gruppen," p. 437.

† That is to say, the composition of transformations generated by the same infinitesimal transformation can always be generated by this infinitesimal transformation.

Groups whose transformations are commutative need not, therefore, be considered in our investigation.

Let us next assume that the symbols of infinitesimal transformations of our group are not in general commutative, but that $(X_j, X_k) = 0$ for $k = 1, 2, \ldots, r$. We shall then have $\left(X_j, \sum_1^r{}^s a_s X_s\right) = 0$ for every system of values of the a's; and, consequently, by theorem I, each transformation generated by X_j will be commutative with every transformation of our group.

Let, now, S_a denote a finite transformation generated by X_j, that is, let S_a be defined by the equations

$$x_i' = x_i + a_j X_j x_i + \frac{1}{1.2}(a_j X_j)^2 x_i + \frac{1}{1.2.3}(a_j X_j)^3 x_i + \ldots.$$
$$(i = 1, 2, \ldots, n).$$

Further, let T_a' and T_b' denote two finite transformations generated respectively by the infinitesimal transformations

$$a_1 X_1 + \ldots a_{j-1} X_{j-1} + a_{j+1} X_{j+1} + \ldots + a_r X_r$$

and $$b_1 X_1 + \ldots b_{j-1} X_{j-1} + b_{j+1} X_{j+1} + \ldots + b_r X_r.$$

If, now, T_a denotes the transformation generated by $a_1 X_1 + \ldots a_{j-1} X_{j-1} + a_j X_j + a_{j+1} X_{j+1} + \ldots a_r X_r$, then, by theorem I,

$$T_a = S_{a_j} T_a' = T_a' S_{a_j}.$$

Similarly, $$T_b = S_{b_j} T_b' = T_b' S_{b_j}.$$

Moreover, $$T_b T_a = S_{a_j} S_{b_j} T_b' T_a' = S_{a_j + b_j} T_b' T_a'.$$

Furthermore, from theorem I, it follows that if $T_b' T_a'$ is non-singular, $S_{a_j+b_j} T_a T_a'$ is non-singular. For $S_{a_j+b_j}$ is generated by the infinitesimal transformation $(a_j + b_j) X_j$, and is, therefore, commutative with every transformation of our group. If, then, $T_b' T_a'$ is generated by the infinitesimal transformation $c_1^{(1)} X_1 + \ldots + c_j^{(1)} X_j + \ldots + c_r^{(1)} X_r$, $S_{a_j+b_j} T_b' T_a'$, by theorem I is generated by the infinitesimal transformation $c_1^{(1)} X_1 + \ldots (c_j^{(1)} + a_j + b_j) X_j + \ldots + c_r^{(1)} X_r$. Consequently, in order that $T_b T_a$ shall be singular, $T_b' T_a'$ must be singular, but $T_b' T_a'$ is independent of a_j and b_j. Whence we obtain

THEOREM III.—*If* T_a *and* T_b, *generated respectively by* $\sum_1^r a_s X_s$ *and* $\sum_1^r b_s X_s$, *are two transformations of the group* $X_1, \ldots . X_j, \ldots . X_r$, *and if* X_j *is commutative with each symbol of infinitesimal transformation, that is, if* $(X_j, X_k) = 0$, $(k = 1, 2, \ldots . , r)$, *then the singularity of the transformation* $T_b\, T_a$ *is independent of the particular values assigned to* a_j *and* b_j.

We may, therefore, let $a_j = b_j = 0$ without affecting our results.

I find it convenient to employ the following notation in dealing with linear homogeneous groups. The transformation defined by the system of equations

$$x_i' = a_{i1}x_1 + a_{i2}x_2 + \ldots . + a_{in}x_n$$
$$(i = 1, 2, \ldots . , n)$$

may be denoted by T_a. The identical transformation defined by

$$x_i' = x_i, \qquad (i = 1, 2, \ldots . n)$$

may be denoted by I. We have $IT = TI$. Further, the transformation defined by

$$x_i' = \sigma x_i, \qquad (i = 1, 2, \ldots . , n)$$

where σ is independent of the x's, may be denoted by σI, or simply by σ; and the composition of T and σ, defined by the equations

$$x_i' = \sigma(a_{i1}x_1 + a_{i2}x_2 + \ldots . a_{in}x_n),$$
$$(i = 1, 2, \ldots . , n)$$

may be denoted by σT. We evidently have

$$\sigma T = T\sigma.$$

There exists a well-known relation between the general projective group of the plane (x, y) and the general linear homogeneous group in three variables, x_1, x_2, x_3.* The nature of this relation is made evident if we consider x_1, x_2, x_3 as homogeneous coordinates of the plane. Then the transformation T of the general linear homogeneous group determines a corresponding projective transformation $\mathbf{T}$ of the plane. To show this, let T be defined by the three equations

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3, \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned}\right\} \qquad (A)$$

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* Lie, "Continuerliche Gruppen." p. 500.

If, now, x'_1, x'_2, x'_3 and x_1, x_2, x_3 be considered as homogeneous coordinates, these equations are equivalent to the two following equations in Cartesian coordinates:

$$\left.\begin{aligned} x' &= \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \\ y' &= \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}, \end{aligned}\right\} \tag{B}$$

which define a projective transformation **T** of the plane (x, y). In the determination of this projective transformation **T**, we are concerned only with the ratios of the a's in the equations defining T. Consequently, the transformations T and σT of the general linear homogeneous group, for every value of σ other than zero, correspond to the same projective transformation **T** of the plane. In other words:* To every transformation **T** of the general projective group in two variables (x, y) is associated ∞^1 transformations σT of the general linear homogeneous group in three variables (x_1, x_2, x_3); and, conversely, to ∞^1 transformations σT of the general linear homogeneous group in three variables is associated a single transformation **T** of the general projective group in two variables.

Further, to every subgroup Γ_r of the general projective group in two variables is associated two or more subgroups G_r, G'_r .... of the general linear homogeneous group in three variables, while to every subgroup G_r of the general linear homogeneous group in three variables is associated but *one* subgroup Γ_r of the general projective group in two variables. If the subgroup G_r of the general linear homogeneous group does not contain singular transformations, the corresponding subgroup Γ_r of the general projective group cannot contain singular transformations. On the other hand, if G_r contains a singular transformation T, it does not necessarily follow that the corresponding transformation **T** of Γ_r is singular. For **T** is associated with ∞^1 transformations σT; and, although T may be a singular transformation of G_r, σT, which is a transformation of at least one of the subgroups G'_r .... associated with Γ_r, may, for some value of σ, be non-singular: in which case **T** is also non-singular. If, however, it is impossible to find a value of σ for which σT is a non-singular transformation of one of the groups G_r, G'_r ...., associated with Γ_r, the transformation **T** is a singular transformation of Γ_r, and, consequently, this group contains singular transformations.

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* Lie, "Continuerliche Gruppen," p. 501.

Let equations (A) (page 70), define the transformations of the subgroup G_r of the general linear homogeneous group. Then equations (B) will define the transformations of the subgroup Γ_r of the general projective group. If, now, G_r contains a singular transformation T, in order to determine whether Γ contains singular transformations, we may ascertain directly whether the transformation $\mathbf{T}$, corresponding to T, is a singular transformation of Γ_r. This method often seems preferable, and, in general at least, presents no difficulty.

Let G_r denote a subgroup of the general linear homogeneous group in n variables. The infinitesimal transformations $X_1, \ldots . X_j, \ldots . X_r$ of G_r satisfy relations of the form*

$$(X_i, X_k) \equiv \sum_1^r {}_s\, c_{iks} X_s, \qquad (i, k = 1, 2, \ldots., r)$$

with constant coefficients c_{iks}. Now, it often happens that the coefficients of one (or more) of the X's, say X_j, is zero in each of these parenthesis-expressions, i. e.,

$$c_{ikj} = 0. \qquad (i, k = 1, 2, \ldots., r)$$

In other words, X_j may not occur in any of the parenthesis-expressions for the X's taken in pairs.

Let us assume that $c_{ikj} = 0$ for $i, k = 1, 2, \ldots., r$. Then

$$Y_1, Y_2, \ldots., Y_r,$$

where

$$Y_k \equiv X_k, \qquad (k \neq j)$$
$$Y_j \equiv X_j + U$$

$\left(U\right.$ denoting the perspective transformation $\left. x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + \ldots. + x_n \frac{\partial}{\partial x_n}\right)$ are infinitesimal transformations of a subgroup G'_r of the general linear homogeneous group in n variables;* and, if

$$(Y_i, Y_k) \equiv \sum_1^r {}_s\, c'_{iks} Y_s,$$

we have $c'_{iks} = c_{iks}$. Therefore, if T_a, T_b, T_c are three transformations of G_r generated respectively by

$$a_1 X_1 + a_2 X_2 + \ldots. + a_r X_r,$$
$$b_1 X_1 + b_2 X_2 + \ldots. + b_r X_r,$$
$$c_1 X_1 + c_2 X_2 + \ldots. + c_r X_r,$$

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* Lie, "Continuerliche Gruppen," p. 391.

and such that $T_b\,T_a = T_c$, the three transformations T_a', T_b', T_c' of G_r' generated respectively by

$$a_1 Y_1 + a_2 Y_2 + \ldots . + a_r Y_r,$$
$$b_1 Y_1 + b_2 Y_2 + \ldots . + b_r Y_r,$$
$$c_1 Y_1 + c_2 Y_2 + \ldots . + c_r Y_r,$$

will satisfy the relation $T_b'\,T_a' = T_c'$, and conversely. Now, the six transformations T_a, T_a', etc., belong to the general linear homogeneous group. Moreover, U is an infinitesimal transformation of the general linear homogeneous group. Therefore, by theorem I, T_a' may be considered as the resultant of the composition of T_a and a transformation generated by $a_j U$. In the notation adopted, the latter is represented by e^{a_j}, since it is defined by the equations

$$x_i' = x_i + a_j U x_i + \ldots . = e^{a_j} x_i. \qquad (i = 1, 2, \ldots . , n)$$

Consequently, $$T_a' = e^{a_j}\,T_a.$$

Similarly, $$T_b' = e^{b_j}\,T_b,$$
$$T_c' = e^{c_j}\,T_c.$$

Therefore, $$e^{c_j}\,T_c = T_c' = T_b'\,T_a' = e^{a_j} e^{b_j}\,T_b\,T_a = e^{a_j + b_j}\,T_c.$$

Whence it follows that

$$e^{c_j} = e^{a_j + b_j};$$

and so we may put $c_j = a_j + b_j$.

We thus obtain the following theorem, which, on account of the close relation existing between the general linear homogeneous group in n variables and the general projective group in $n - 1$ variables, applies also to the general projective group in $(n - 1)$ variables:

THEOREM IV.—*If T_a, T_b, and T_c, generated respectively by $\sum_1^r{}_s\, a_s X_s$, $\sum_1^r{}_s\, b_s X_s$, and $\sum_1^r{}_s\, c_s X_s$, and connected by the relation $T_b\,T_a = T_c$, are three transformations of the group $X_1, \ldots . X_j, \ldots . X_r$; and if, moreover, in the parenthesis-expressions*

$$(X_i, X_k) \equiv \sum_1^r{}_s\, c_{iks} X_s, \qquad (i, k = 1, 2, \ldots . , r)$$

the coefficients of X_j *are zero, that is, if*

$$c_{ikj} = 0\,, \qquad (i,\, k = 1,\, 2,\, \ldots\ldots,\, r)$$

we may then put $c_j = a_j + b_j$.

In the case of the linear homogeneous groups, the process of finding the equations of the finite transformations of a given group, $X_1, \ldots, X_r$, can often be very greatly simplified. For let T be a transformation of G_r generated by the infinitesimal transformation $\sum_{1}^{r}{}_{s}\, l_s X_s$. To the symbol $\sum_{1}^{r}{}_{s}\, l_s X_s$ add and subtract the same multiple of U, where U denotes the perspective transformation,

$$x_1 \frac{\partial}{\partial x_1} + \ldots\ldots + x_n \frac{\partial}{\partial x_n}\,.$$

Then, after rearranging and simplifying the terms, le

$$\sum_{1}^{r}{}_{s}\, l_s X_s \equiv \overline{X} + \varepsilon U,$$

where $\overline{X}$ is the symbol of an infinitesimal transformation, namely, $\sum_{1}^{r}{}_{s}\, l_s X_s - \varepsilon U$. Whether $\overline{X}$ is a transformation of the group G_r is immaterial; it is sufficient that $\overline{X}$ is an infinitesimal transformation of the general linear homogeneous group, of which G_r is a subgroup. Since

$$(\overline{X},\, \varepsilon U) \equiv 0\,,$$

we may, by theorem I, consider T as the resultant of the two commutative transformations generated respectively by $\overline{X}$ and εU. According to our notation, the transformation generated by εU may be represented by e^{ε}, since it is defined by the system of equations

$$x_i' = e^{\varepsilon} x_i\,. \qquad (i = 1,\, 2,\, \ldots\ldots,\, n)$$

Consequently, if S be the transformation generated by $\overline{X}$, the transformation T is identical with the transformation $e^{\varepsilon} S$, or

$$T = e^{\varepsilon} S.$$

In other words, if we know the system of equations defining S, we multiply the right-hand members of these equations by the quantity e^{ε} in order to obtain the system of equations that define the transformation T. By a proper choice of ε, $\overline{X}$ often takes such a form that it is much simpler first to find the transformation S than to obtain the transformation T directly. For example, suppose we wish to find the finite transformations generated by the infinitesimal transformations of the two-parameter group, xq, $xp + \alpha yq$. Here

$$\sum_1^r l_s X_s \equiv l_1 xq + l_2 (xp + \alpha yq) \equiv l_1 xq + l_2 (\alpha - 1) yq + l_2 U;$$

and we have $\overline{X} \equiv l_1 xq + l_2 (\alpha - 1) yq$, and $\varepsilon U \equiv l_2 U$. The transformation S generated by $\overline{X}$ is defined by the infinite series

$$\begin{aligned} x' &= x + [l_1 xq + l_2 (\alpha - 1) yq]\, x + \ldots\ldots; \\ y' &= y + [l_1 xq + l_2 (\alpha - 1) yq]\, y + \ldots\ldots, \end{aligned}$$

and can readily be summed. The transformation S multiplied by e^{l_2} gives us the required transformation T. The transformation T, considered as generated by $l_1 xq + l_2 (xp + \alpha yq)$, is defined by the infinite series

$$\begin{aligned} x_1 &= x + [l_1 xq + l_2 (xp + \alpha yq)]\, x + \ldots\ldots, \\ y_1 &= y + [l_1 xq + l_2 (xp + \alpha yq)]\, y + \ldots\ldots \end{aligned}$$

These series are, however, more difficult to sum than the series defining S.

At the end of this paper, a list is given of those two- and three-parameter subgroups of the general projective group in two variables, and of the general linear homogeneous group in three variables, that contain singular transformations, and are, therefore, not properly continuous. They are included among the subgroups of the general projective group and of the general linear homogeneous group enumerated by Lie on pages 288 and 519 respectively of his "Continuerliche Gruppen." The following groups have been selected from this list.

Example 1. The subgroup, xq, $xp + \alpha yq$ $(\alpha \neq 0, 1)$. For this group $\sum_1^r l_s X_s \equiv \overline{X} + \varepsilon U \equiv l_1 xq + l_2 (\alpha - 1) yq + l_2 U$, and equations (3) becomes

$$\begin{aligned} x' &= e^{l_2} x, \\ y' &= \frac{l_1 e^{l_2}}{l_2 (\alpha - 1)} \left(e^{l_2 (\alpha - 1)} - 1\right) x + e^{l_2 \alpha} y. \end{aligned}$$

Consequently, T_a and T_b are defined respectively by the equations

$$\left.\begin{aligned} x' &= e^{a_2} x, \\ y' &= \frac{a_1 e^{a_2}}{a_2(\alpha-1)}\left(e^{a_2(\alpha-1)}-1\right)x + e^{a_2\alpha} y, \end{aligned}\right\} \tag{A}$$

and

$$\left.\begin{aligned} x'' &= e^{b_2} x', \\ y'' &= \frac{b_1 e^{b_2}}{b_2(\alpha-1)}\left(e^{b_2(\alpha-1)}-1\right)x' + e^{b_2\alpha} y'. \end{aligned}\right\} \tag{B}$$

Eliminating x' and y' between equations (A) and (B), we obtain, as the system of equations defining $T_b\, T_a$,

$$\left.\begin{aligned} x'' &= e^{a_2+b_2} x, \\ y'' &= Nx + e^{(a_2+b_2)\alpha} y, \end{aligned}\right\} \tag{C}$$

where N denotes

$$\frac{1}{\alpha-1}\left\{\frac{b_1 e^{b_2}}{b_2}\left(e^{b_2(\alpha-1)}-1\right)e^{a_2} + \frac{a_1 e^{a_2}}{a_2}\left(e^{a_2(\alpha-1)}-1\right)e^{b_2\alpha}\right\}.$$

Moreover, T_c is defined by the system of equations

$$\left.\begin{aligned} x'' &= e^{c_2} x, \\ y'' &= \frac{c_1 e^{c_2}}{c_2(\alpha-1)}\left(e^{c_2(\alpha-1)}-1\right)x + e^{c_2\alpha} y. \end{aligned}\right\} \tag{D}$$

Wherefore, if $T_b\, T_a = T_c$,

$$\begin{aligned} e^{c_2} &= e^{a_2+b_2}, \\ e^{c_2\alpha} &= e^{(a_2+b_2)\alpha}, \\ \frac{c_1 e^{c_2}}{c_2(\alpha-1)}&\left(e^{c_2(\alpha-1)}-1\right) = N. \end{aligned}$$

From the first two of these equations, we obtain

$$\begin{aligned} c_2 &= a_2 + b_2 + 2n\pi i, \\ c_2\alpha &= (a_2+b_2)\alpha + 2n'\pi i, \end{aligned}$$

where n and n' are integers. Whence we have as a condition to which the integers n and n' are subject,

$$n\alpha = n'.$$

If α is irrational, this relation is possible only if both n and n' are zero. In this case equations (2) become

$$\left.\begin{aligned} c_2 &= a_2 + b_2, \\ c_1 &= \frac{(a_2 + b_2)(\alpha - 1)N}{e^{a_2+b_2}\left(e^{(a_2+b_2))(\alpha-1)} - 1\right)} \,. \end{aligned}\right\} \qquad \text{(E)}$$

Whence it follows that c_2 is finite for every set of finite values of the a's and b's; but if $a_2 + b_2 = \frac{2m\pi i}{\alpha - 1}$ (m being an integer $\neq 0$), c_1 will be infinite, provided the a's and b's are so chosen that $N \neq 0$.

Therefore, if α is irrational, the transformation $T_b\, T_a$ is singular, provided the a's and b's are so chosen that $N \neq 0$ and that $a_2 + b_2$ shall be an even multiple of $\pi\iota$ other than zero. Substituting $\frac{2m\pi i}{\alpha - 1}$ in (C) for $a_2 + b_2$, we have, as the equations of definition of the singular transformation $T_b\, T_a$,

$$\begin{aligned} x'' &= e^{\frac{2m\pi i}{\alpha - 1}} x, \\ y'' &= Nx + e^{\frac{2m\alpha\pi i}{\alpha - 1}} y. \end{aligned} \qquad (N \neq 0,\ m \text{ an integer } \neq 0)$$

Let α be rational. In this case n is subject only to the condition that $n\alpha$ shall be an integer; and, if we put $\alpha = \frac{\mu}{\nu}$, where μ and ν are two integers relatively prime, $n = \lambda\nu$, where λ is an arbitrary integer. We now have

$$\begin{aligned} c_2(\alpha - 1) &= (a_2 + b_2 + 2n\pi i)(\alpha - 1) \\ &= (a_2 + b_2)\frac{\mu - \nu}{\nu} + 2\lambda(\mu - \nu)\pi i. \end{aligned}$$

And, therefore, equations (2) become

$$\begin{aligned} c_2 &= a_2 + b_2 + 2\lambda\nu\pi i, \\ c_1 &= \frac{\left[(a_2 + b_2)\frac{\mu - \nu}{\nu} + 2\lambda(\mu - \nu)\pi i\right] N}{e^{a_2+b_2}\left(e^{(a_2+b_2)\frac{\mu-\nu}{\nu}} - 1\right)} \,. \end{aligned}$$

As before, c_2 is finite for finite values of the a's and b's. But, if $a_2 + b_2 = \frac{2m\nu\pi i}{\mu - \nu}$ (m an integer), c_1 is infinite, provided $N \neq 0$, unless we can so choose λ that

$$\lambda(\mu - \nu) + m = 0;$$

in which case $c_1 = N$.

Consequently, if α is rational and equal to $\frac{\mu}{\nu}$, where μ and ν are two integers relatively prime, the transformation $T_b\,T_a$ defined by the equations

$$x' = e^{\frac{2m\nu\pi i}{\mu-\nu}}\,x,$$

$$y' = Nx + e^{\frac{2m\mu\pi i}{\mu-\nu}}\,y, \qquad (N \neq 0)$$

is singular, unless m contains $\mu - \nu$, in which case $T_b\,T_a$ is non-singular.

Let α be rational and equal to $\frac{\mu}{\nu}$; and let the a's and b's be so chosen that $T_b\,T_a$ is singular. Apply the transformation $T_b\,T_a$ successively $|\mu - \nu|$ times to the manifold (x, y). The resulting transformation is defined by the equations

$$x' = x,$$

$$y' = N'x + y,$$

where, denoting $e^{\frac{2m\nu\pi i}{\mu-\nu}}$ and $e^{\frac{2m\mu\pi i}{\mu-\nu}}$ respectively by ρ and σ, $N' = N(\rho^{\mu-\nu-1} + \rho^{\mu-\nu-2}\sigma + \ldots\ldots + \rho\sigma^{\mu-\nu-2} + \sigma^{\mu-\nu-1})$. This transformation is, however, non-singular, and can be generated by the infinitesimal transformation $N'q + 0\,(xp + \alpha yq)$, as may be seen on substituting in equation (D) $c_1 = N'$ and $c_2 = 0$. Thus, although $T_b\,T_a$ may be singular, if α is rational, some power of $T_b\,T_a$ is non-singular.* The significance of this fact will appear later.

Example 2. $xq,\; xp + q$.

The transformation, T_c, is defined by the equations

$$\left.\begin{aligned} x' &= e^{c_2}\,x, \\ y' &= \frac{c_1}{c_2}\,(e^{c_2} - 1)\,x + y + c_2. \end{aligned}\right\} \qquad \text{(A)}$$

And, if $T_b\,T_a = T_c$, we have also

$$\left.\begin{aligned} x' &= e^{a_2 + b_2}\,x, \\ y' &= Nx + y + (a_2 + b_2), \end{aligned}\right\} \qquad \text{(B)}$$

where N is a function of the a's and b's and may be taken arbitrarily.

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* Cf. Study, Leipzige Berichte, 1892. Taber, Proc. Lond. Math. Soc., vol. XXVI; Math. Ann., vol. XLVI; Bul. N. Y. Math. Soc., July, 1894.

Equations (B) may be obtained either in a manner similar to that in which equations (C) in example 1 were obtained, or they may be obtained immediately by means of theorem IV. We have $(xq, xp+q) = -xq$; and, consequently, if $T_b T_a = T_c$, we may put $c_2 = a_2 + b_2$. We may assume, therefore, that $T_b T_a$ is defined by equations (B) where N is a function of the a's and b's, and may be taken arbitrarily. In most cases I found the latter method preferable.

Comparing the two sets of equations (A) and (B), we obtain

$$c_2 = a_2 + b_2,$$
$$c_1 = \frac{N(a_2+b_2)}{e^{a_2+b_2}-1}.$$

From these it follows that every branch of c_1 is infinite for $a_2 + b_2 = 2m\pi i$ (m being an integer $\neq 0$), provided the a's and b's are so chosen that $N \neq 0$. Consequently, the given group contains singular transformations.

Example 3. $\quad x_3p_2,\ x_1p_2,\ x_3p_1 + x_2p_2 + \alpha U.$

Here $\sum_1^r l_s X_s \equiv \bar{X} + \varepsilon U \equiv l_1 x_3 p_2 + l_2 x_1 p_2 + l_3(x_3 p_1 + x_2 p_2) + l_3 \alpha U$, and the transformation T_c is defined by the equations

$$\begin{aligned} x_1' &= e^{c_3\alpha} x_1 + c_3 e^{c_3\alpha} x_3, \\ x_2' &= \frac{c_2 e^{c_3\alpha}}{c_3}(e^{c_3}-1)x_1 + e^{c_3(\alpha+1)}x_2 + e^{c_3\alpha}\left(\frac{c_1+c_2}{c_3}(e^{c_3}-1) - c_2\right)x_3, \\ x_3' &= e^{c_3\alpha} x_3. \end{aligned}$$

If $T_c = T_b T_a$, T_c is defined also by the equations

$$\begin{aligned} x_1' &= e^{(a_3+b_3)\alpha} x_1 + (a_3+b_3) e^{(a_3+b_3)\alpha} x_3, \\ x_2' &= Nx_1 + e^{(a_3+b_3)(\alpha+1)} x_2 + Mx_3, \\ x_3' &= e^{(a_3+b_3)\alpha} x_3, \end{aligned}$$

where M and N are functions of the a's and b's and may be taken arbitrarily. Wherefore,

$$\begin{aligned} e^{c_3\alpha} &= e^{(a_3+b_3)\alpha}, \\ c_3 &= a_3 + b_3, \\ e^{c_3\alpha}\frac{c_2}{c_3}(e^{c_3}-1) &= N, \end{aligned}$$
$$e^{c_3\alpha}\frac{c_1}{c_3}(e^{c_3}-1) + \frac{e^{c_3\alpha}c_2}{c_3}(e^{c_3}-1) - c_2 e^{c_3\alpha} = M.$$

Whence,

$$\begin{aligned} c_3 &= a_3 + b_3, \\ c_2 &= \frac{N(a_3 + b_3)}{e^{(a_3+b_3)\alpha}\left(e^{a_3+b_3} - 1\right)}, \\ c_1 &= \frac{M - N + c_2\, e^{(a_3+b_3)\alpha}}{e^{(a_3+b_3)\alpha}\left(e^{a_3+b_3} - 1\right)}\,(a_3 + b_3). \end{aligned}$$

From these equations it follows that every branch of both c_2 and c_1 is infinite if $a_3 + b_3 = 2m\pi i$ (m being an integer $\neq 0$), provided $N \neq 0$. Moreover, c_1 is then infinite of an order higher than c_2; and if $N = 0$, c_1 is infinite, provided $M + c_2\, e^{2m\alpha\pi i} \neq 0$.

Consequently, the given group contains singular transformations. The group associated with this group, namely, q, xq, $p + yq$, likewise contains singular transformations.

Example 4.—The subgroup, $x_3 p_1$, $x_3 p_2$, $\alpha x_1 p_1 + \beta x_2 p_2 + \vartheta x_3 p_3$. The transformation T_c is defined by the equations

$$\begin{aligned} x_1' &= e^{c_3\alpha}\, x_1 + \frac{c_1\, e^{c_3\vartheta}}{c_3(\alpha - \vartheta)}\left(e^{c_3(\alpha-\vartheta)} - 1\right) x_3, \\ x_2' &= e^{c_3\beta}\, x_2 + \frac{c_2\, e^{c_3\vartheta}}{c_3(\beta - \vartheta)}\left(e^{c_3(\beta-\vartheta)} - 1\right) x_3, \\ x_3' &= e^{c_3\vartheta}\, x_3. \end{aligned}$$

If $T_c = T_b\, T_a$, T_c is also defined by

$$\begin{aligned} x_1' &= e^{(a_3+b_3)\alpha}\, x_1 + M x_3, \\ x_2' &= e^{(a_3+b_3)\beta}\, x_2 + N x_3, \\ x_3' &= e^{(a_3+b_3)\vartheta}\, x_3. \end{aligned}$$

Wherefore,

$$\begin{aligned} e^{c_3\alpha} &= e^{(a_3+b_3)\alpha}, \\ e^{c_3\beta} &= e^{(a_3+b_3)\beta}, \\ e^{c_3\vartheta} &= e^{(a_3+b_3)\vartheta}, \\ \frac{c_1\, e^{c_3\vartheta}}{c_3(\alpha - \vartheta)}&\left(e^{c_3(\alpha-\vartheta)} - 1\right) = M, \\ \frac{c_2\, e^{c_3\vartheta}}{c_3(\beta - \vartheta)}&\left(e^{c_3(\beta-\vartheta)} - 1\right) = N. \end{aligned}$$

From the first three of these equations, we obtain

$$\begin{aligned} c_3\,\alpha &= (a_3 + b_3)\,\alpha + 2n\pi i, \\ c_3\,\beta &= (a_3 + b_3)\,\beta + 2n'\pi i, \\ c_3\,\vartheta &= (a_3 + b_3)\,\vartheta + 2n''\pi i, \end{aligned}$$

where n, n' and n'' are integers. Since these equations are simultaneous, we have the relations

$$\frac{n}{\alpha}=\frac{n'}{\beta}=\frac{n''}{\vartheta}\equiv\rho.$$

For certain values of α, β and ϑ (e. g., if α is rational, β and ϑ irrational), these relations are possible only if n, n' and n'', and therefore ρ are zero. With the foregoing meaning assigned to ρ, equations (2) become

$$\begin{aligned} c_3 &= a_3+b_3+2\rho\pi i, \\ c_2 &= \frac{(\beta-\vartheta)N(a_3+b_3+2\rho\pi i)}{(e^{(a_3+b_3)\vartheta}(e^{(a_3+b_3)(\beta-\vartheta)}-1)} \\ c_1 &= \frac{(\alpha-\vartheta)M(a_3+b_3+2\rho\pi i)}{e^{(a_3+b_3)\vartheta}(e^{(a_3+b_3)(\alpha-\vartheta)}-1)} \end{aligned}$$

From these equations it follows that every branch of c_2 is infinite if $a_3+b_3=\frac{2m\pi i}{\beta-\vartheta}$ (provided $N\neq 0$), unless $\rho+\frac{m}{\beta-\vartheta}=0$, in which case c_2 is finite. But if $\rho=-\frac{m}{\beta-\vartheta}$, $\frac{m\alpha}{\beta-\vartheta}$, $\frac{m\beta}{\beta-\vartheta}$, and $\frac{m\vartheta}{\beta-\vartheta}$ are all integers. Consequently, if $a_3+b_3=\frac{2m\pi i}{\beta-\vartheta}$, and if, moreover, $\frac{m\alpha}{\beta-\vartheta}$, $\frac{m\beta}{\beta-\vartheta}$ and $\frac{m\vartheta}{\beta-\vartheta}$ are not all integers, every branch of c_2 is infinite (provided $N\neq 0$), and the corresponding transformation is singular. Similarly, every branch of c_1 is infinite if $a_3+b_3=\frac{2m\pi i}{\alpha-\vartheta}$, and if, moreover, $\frac{m\alpha}{\alpha-\vartheta}$, $\frac{m\beta}{\alpha-\vartheta}$, and $\frac{m\vartheta}{\alpha-\vartheta}$, are not all integers and $M\neq 0$.

Example 5.—The special linear homogeneous group

$$x_1p_2,\ x_1p_1-x_2p_2,\ x_2p_1,$$

The transformation T_c is defined by the equations

$$\begin{aligned} x_1' &= \left[\frac{e^d+e^{-d}}{2}+\frac{c_2}{d}\left(\frac{e^d-e^{-d}}{2}\right)\right]x_1+\frac{c_3}{d}\left(\frac{e^d-e^{-d}}{2}\right)x_2, \\ x_2' &= \frac{c_1}{d}\left(\frac{e^d-e^{-d}}{2}\right)x_1+\left[\frac{e^d+e^{-d}}{2}-\frac{c_2}{d}\left(\frac{e^d-e^{-d}}{2}\right)\right]x_2, \\ x_3' &= x_3, \end{aligned}$$

where $d=\sqrt{c_2^2+c_1c_3}$.

Let $T_b\, T_a$ be defined by the equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2, \\ x_2' &= a_{21}x_1 + a_{22}x_2, \\ x_3' &= x_3. \end{aligned} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1,$$

Then, if $T_b\, T_a = T_c$,

$$e^d + e^{-d} = a_{11} + a_{22}, \tag{A}$$

$$c_1\left(\frac{e^d - e^{-d}}{d}\right) = 2a_{21}, \tag{B}$$

$$c_2\left(\frac{e^d - e^{-d}}{d}\right) = a_{11} - a_{22}, \tag{C}$$

$$c_3\left(\frac{e^d - e^{-d}}{d}\right) = 2a_{12}, \tag{D}$$

where

$$a_{11}a_{22} - a_{12}a_{21} = 1 \tag{E}$$

and

$$d = \sqrt{c_2^2 + c_1c_3}. \tag{F}$$

By means of equations (B), (C) and (D), we obtain the equation

$$(c_2^2 + c_1c_3)\left(\frac{e^d - e^{-d}}{d}\right)^2 = (a_{11} - a_{22})^2 + 4a_{21}a_{12},$$

whence

$$(e^d - e^{-d})^2 = (a_{11} + a_{22})^2 - 4, \qquad \text{(by (E}$$

or

$$e^d - e^{-d} = \pm\sqrt{(a_{11} + a_{22})^2 - 4}.$$

Adding (A) and (G),

$$e^d = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4}}{2}.$$

Taking the logarithms of both sides,

$$d = \log\frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4}}{2} + 2k\pi i, \tag{H}$$

where k is an integer.

Since a_{11} and a_{22} are functions of the a's and b's, d is a multivalued function of the a's and b's.

Moreover, from equations (B), (C) and (D) we obtain

$$c_1 = \frac{2a_{21}}{e^d - e^{-d}} d,$$

$$c_2 = \frac{a_{11} - a_{22}}{e^d - e^{-d}} d,$$

$$c_3 = \frac{2a_{12}}{e^d - e^{-d}} d,$$

where d is defined by (H).

From (H), it follows that $d = 2k\pi i$, if $a_{11} + a_{22} = 2$. But if $d = 2k\pi i$, c_1 will be infinite (provided $a_{21} \neq 0$), unless we put $k = 0$, in which case $c_1 = a_{21}$, and, then, at least one branch of c_1 is finite. Similarly, at least one branch of c_2 and c_3 respectively is finite. Consequently, if $a_{11} + a_{22} = 2$, the transformation $T_b\, T_a$ is non-singular.

On the other hand, if $a_{11} + a_{22} + 2 = 0$, $d = (2k + 1)\pi i$. But in this case, c_1 will be infinite for any integer k, provided $a_{21} \neq 0$; and consequently, if $a_{21} \neq 0$, every branch of c_1 will be infinite. Similarly, every branch of c_2 will be infinite, provided $a_{11} \neq a_{22}$; and every branch of c_3 will be infinite, provided $a_{12} \neq 0$.

Let $a_{11} + a_{22} = -2$. Squaring,

$$a_{11}^2 + a_{22}^2 + 2a_{11}\, a_{22} = 4.$$

From (E),
$$4a_{11}\, a_{22} = 4 + 4a_{12}\, a_{21}.$$

Subtracting,
$$(a_{11} - a_{22})^2 = -4a_{12}\, a_{21}.$$

If, therefore, a_{12} or a_{21} is zero, $a_{11} = a_{22}$, and c_2 will not be infinite. Consequently, *if* $a_{11} + a_{22} + 2 = 0$, *and if, moreover,* a_{12} *and* a_{21} *are not both zero, the transformation* $T_b\, T_a$ *is singular.*

We know that a point p of general position in the space $(x_1 \ldots . x_n)$ will describe a continuous curve under the successive application of the infinitesimal transformation that generates a given transformation T. This continuous curve is called the *path-curve of the point p with respect to, or associated with, the infinitesimal transformation* in question, and is invariant with respect to this infinitesimal transformation, and, therefore, invariant with respect to the transformation T. It is obvious that there are ∞^{n-1} path-curves associated with any infinitesimal transformation that generates T (or ∞^{n-2}, if the space $(x_1 \dots x_n)$ is $(n-1)$-way extended).*

Let $X_1, X_2, \dots, X_r$ be the infinitesimal transformations of the group G_r. Let T be a transformation of this group, and

$$\sum_1^r{}^j c_j X_j = \sum_1^n{}^s \xi_s (x_1 \dots x_n, c_1 \dots c_r) \frac{\partial}{\partial x_s}$$

be the infinitesimal transformation that generates T. The path-curve of the point p with respect to this infinitesimal transformation may be termed the *path-curve of the point p associated with the transformation T*. The invariant curves, or path-curves, associated with T are given, therefore, by the system of simultaneous differential equations,†

$$\frac{dx_1}{\xi_1(x, c)} = \frac{dx_2}{\xi_2(x, c)} = \dots = \frac{dx_n}{\xi_n(x, c)}. \tag{A}$$

Let the $(n-1)$ independent integrals be

$$\left.\begin{array}{l} \phi_1 \;\;(x_1 \dots x_n, c_1 \dots c_r) = K_1, \\ \phi_2 \;\;(x_1 \dots x_n, c_1 \dots c_r) = K_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \phi_{n-1}(x_1 \dots x_n, c_1 \dots c_r) = K_{n-1}. \end{array}\right\} \tag{B}$$

The equations (B), by giving all possible values to the constants of integration, $K_1 \dots K_{n-1}$, define the ∞^{n-1} of invariant curves, or path-curves associated with T. But when T is singular, no infinitesimal transformation of the group to which T belongs will generate T, and, therefore, there is no path-curve associated with T.‡ Nevertheless, there are always, even in this case, curves invariant to T.

If we let $c_1 \dots c_r$ take all possible sets of values, equations (B) define the path-curves or invariant curves associated with the transformations of the given group G_r. In the case of a singular transformation, one (or more) of the parameters, say c_j, is infinite. Suppose $\phi_i = K_i$ contains the parameter c_j, then if c_j is put infinite in this equation, certain of the terms in ϕ_i may be infinite.

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* Lie, "Differential Gleichungen," p. 300.

† Lie, Ibid., p. 301.

‡ That is, no one of the path-curves generated by infinitesimal transformations of G_r is associated with T.

If the highest order of infinity of the terms thus rendered infinite is q, we may choose K_i infinite also of the q^{th} order. If, then, we divide through by a quantity infinite of the q^{th} order, the terms finite, or infinite, of an order lower than the q^{th}, will disappear. Proceeding similarly with the other equations, we obtain equations defining the curves invariant to those singular transformations of the given group for which $c_j = \infty$. If the equations (B) are all algebraic, that is, if all the path-curves or invariant curves associated with the infinitesimal transformations of G_r are algebraic, each of the curves invariant to a singular transformation T, in such cases as I have examined, break up into a finite system of lines (real or imaginary); and the application of the singular transformation T to a point p on one of these lines (not itself invariant) will convey p from the line in question to another of the same system. Moreover, the transformation repeated a certain number of times will convey the given point p back to the original line. Here we have a geometrical reason why some power of a singular transformation may often be non-singular; namely, some power of a singular transformation T applied to a point p on one of the lines of the system invariant, as a whole, to T, may move the given point along that line, which transformation may be effected by the repetition of an infinitesimal transformation of G_r.

On the other hand, if the path-curves, or invariant curves (or at least some of them), are transcendental, it often happens that each of the curves invariant to a singular transformation consists of an infinite number of lines; and the application of the singular transformation to a point p on one of these lines (not itself invariant) will convey p from that line to another line of the system; but, repeated any number of times, it will not, as in the preceding case, convey p back to the original line. In some cases each curve invariant to a singular transformation may even reduce to a single line.

I shall investigate the invariant curves of the first three groups considered above.

Example 1.—The invariant curves (path-curves) associated with the transformations of the group xq, $xp + ayq$ are given by the differential equation

$$\frac{dx}{c_2 x} = \frac{dy}{c_1 x + c_2 a y}. \tag{A}$$

Integrating, and denoting the constant of integration by K, we obtain the equation

$$x^{\alpha} = \frac{c_1}{K}x + \frac{c_2}{K}(\alpha - 1)y, \tag{B}$$

which, by giving proper values to c_1 and c_2, defines the curves invariant to the transformation T of the given group. If, now, T is singular, then c_1 is infinite (see p. 76 *et seq.*); but since K is an arbitrary constant, we may choose K so that $\frac{c_1}{K} = C$, where C is finite, in which case $\frac{c_2}{K}$ is zero. The equation (B) thus becomes

$$x^{\alpha} = Cx, \tag{C}$$

which represents a system of lines invariant as a whole to T.* The application of the singular transformation T to a point p of general portion on one of these lines (not itself invariant to T) will convey p from that line to another line of the system. For example (see p. 78), let $m = 1$ and $\alpha = 3$. Then the singular transformation T is defined by

$$\begin{aligned} x' &= -x, \\ y' &= Nx - y, \end{aligned} \qquad (N \neq 0)$$

and the equation of the invariant curves reduces to

$$x^3 = Cx,$$

which breaks up into

$$x = +C, \quad x = -C,$$

and the special invariant $x = 0$.

Moreover, the application of T to a point p of general position on the line $x = +C$ will convey that point to the line $x = -C$. Applied again, it will reconvey the point p to the line $x = +C$. The second power of T, therefore, moves the point p along the line $x = +C$; and this transformation may be effected continuously by the repetition of an infinitesimal transformation of our group; that is to say, T^2 is non-singular.

---

* Each of the lines of this system is separately invariant to an infinitesimal transformation of G_r; namely, that for which $c_1 = 1$, $c_2 = 0$. For the path-curves of this infinitesimal transformation are defined by the equation $x =$ constant.

Example 2.—The invariant curves of the group xq, $xp + q$ are given by the differential equation

$$\frac{dx}{c_2 x} = \frac{dy}{c_1 x + c_2}. \tag{A}$$

Integrating, and denoting the constant of integration by K, we obtain

$$c_2 y = c_1 x + c_2 \log x + K. \tag{B}$$

For a singular transformation, c_1 is infinite (see p. 79). Choosing K properly, equation (B) becomes

$$x = C. \tag{C}$$

Each curve invariant to the transformation T reduces, therefore, to a single line, if T is singular.

Example 3.—The invariant curves of the group $x_3 p_2$, $x_1 p_2$, $x_3 p_1 + x_2 p_2$ are given by the simultaneous differential equations

$$\frac{dx_1}{c_3 x_3} = \frac{dx_2}{c_1 x_3 + c_2 x_1 + c_3 x_2} = \frac{dx_3}{0}. \tag{A}$$

Integrating, and denoting the constants of integration by K and C, we obtain the equations

$$\begin{aligned} e^{-\frac{x_1}{x_3}} x_2 &= -\frac{c_2}{c_3} x_3 e^{-\frac{x_1}{x_3}} \left(\frac{x_1}{x_3} + 1\right) - \frac{c_1}{c_3} x_3 e^{-\frac{x_1}{x_3}} + K, \\ x_3 &= C. \end{aligned} \tag{B}$$

For a singular transformation, c_1 is infinite of an order higher than c_2. Consequently, equations (B) reduce to

$$\begin{aligned} e^{\frac{x_1}{x_3}} &= K' x_3, \\ x_3 &= C, \end{aligned}$$

from which we obtain

$$\begin{aligned} \frac{x_1}{x_3} &= C' + 2k\pi i, \\ x_3 &= C; \end{aligned}$$

or

$$\begin{aligned} &x_1 = x_3 C', \quad x_3 = C, \\ &x_1 = x_3 (C' + 2\pi i), \quad x_3 = C, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &x_1' = x_3 (C' + 2m\pi i), \quad x_3 = C, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Consequently, if T is singular, each curve invariant to T consists of an infinite number of lines. Moreover, the singular transformation applied to a point p on one of these lines (not itself invariant) will convey p from that line to another of the system.

The singular transformation is defined by the equations

$$\begin{aligned} x_1' &= x_1 + 2m\pi i x_3, \\ x_2' &= Nx_1 + x_2 + Mx_3, \\ x_3' &= x_3. \end{aligned} \qquad (N \neq 0)$$

This transformation applied to the point (CC', x_2, C) on the line

$$x_1 = x_3 C', \quad x_3 = C$$

will convey that point to the point $(CC' + 2m\pi i C, x_2', C)$ which, evidently, is a point on the line

$$x_1 = x_3 (C' + 2m\pi i), \quad x_3 = C.$$

Two points, p and p_1, of general position on the same smallest invariant manifold relative to the group G_r can always be interchanged by one or more transformations of G_r. In general, each of the transformations by which p and p_1 can be interchanged, can be generated by an infinitesimal transformation of G_r; in which case I shall say that the points p and p_1 can be continuously interchanged by the transformations of this group. But if G_r contains singular transformations, it sometimes happens that the points p and p_1 cannot be interchanged by a transformation of G_r that can be generated by an infinitesimal transformation of G_r; and, in this case, I shall say that the points p and p_1 cannot be continuously interchanged.

If the smallest invariant manifold relative to the r-parameter group G_r is q-way extended, $q \leqq r$, then there are ∞^{r-q} transformations of G_r that will interchange two points, p and p_1 of general position on any invariant manifold relative to G_r. If $r = q$, then there is at least *one* transformation, and in the case of certain groups there may be ∞^1.* If there is but one, and if this transformation is singular, that is, if this transformation cannot be generated by an infinitesimal transformation of G_r, then, clearly, not all points on each smallest invariant manifold can be continuously interchanged. But if there are more transformations than one that will interchange two points p and p_1 of general position, then it is by no means certain, when G_r contains singular transformations, that p and p_1 can be chosen so that all these transformations are singular; or, if $q < r$, that the ∞^{r-q} transformations are singular. In fact, in all cases I have considered, this is never possible. It may happen that but one, or all but one, of these transformations are singular. In this case the points of general position on any smallest invariant manifold can be continuously interchanged by means of the transformation of the given group, although the group contain transformations that cannot be generated by an infinitesimal transformation of the group.

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* Lie, "Continuerliche Gruppen," p. 432, Lie here states that there is but one transformation if the group is simply transitive. But this is not always true, as is shown by the 2nd example below.

Most of the conclusions and examples which follow have been given by the author in a "Note on the Projective Group." Proc. Amer. Acad., vol. XXXIII, p. 493.

Associated with a group G_r with r-parameters is a subgroup $\mathbf{G}_\rho$ with $\rho \leqq r$ parameters of the general linear homogeneous group in r-variables—the adjoined of the group G_r. If these r-variables be considered as Cartesian coordinates, then every point in the space, S_r, to which the transformations of $\mathbf{G}_r$ are applied, represents a transformation of the group G_r. On the other hand, if the r-variables be considered as homogeneous coordinates, then every point in the space S_{r-1}, to which the transformations of $\mathbf{G}_\rho$ are applied, represents a one-parameter subgroup of G_r. Moreover, two points p and p_1 of general position on the same smallest invariant manifold in S_r (or S_{r-1}) relative to $\mathbf{G}_\rho$, represent transformations (or one-parameter subgroups) of G_r of the same type. For these points can be interchanged by means of one or more transformations of $\mathbf{G}_\rho$; and, therefore, the transformations (or one-parameter subgroups) represented by these points, may be transformed into each other by means of the transformations of G_r. If, however, the points p and p_1 cannot be continuously interchanged, the transformations (or one-parameter subgroups) represented by these points, although of the same type, are differently related from two transformations (or one-parameter subgroup) represented by points in S_r (or S_{r-1}) that can be continuously interchanged.*

I have examined all the two- and three-parameter groups enumerated by Lie in the "Continuerliche Gruppen," pp. 288 and 519. In each case the adjoined group $\mathbf{G}_\rho$ is such that two points of general position on the same smallest invariant manifold relative to $\mathbf{G}_\rho$ can always be interchanged continuously, notwithstanding that in certain cases the associated group $\mathbf{G}_\rho$ contains singular transformations. I have, therefore, as yet, found no group G_r whose transformations (or one-parameter subgroups) of the same type cannot be continuously transformed into each other by means of transformations of G_r; and it seems by no means certain that such groups exist.

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* Proc. Am. Acad., vol. XXXIII, p. 493.

Let S and T be two transformations of the group G_r, and let S be transformed into S' by means of the transformation T, that is, let

$$TST^{-1} = S'.$$

If T is non-singular, S may be continuously transformed into S' by means of the infinitesimal transformation of G_r that generates T. But if T is singular, there is no infinitesimal transformation of G_r that generates T. Now, in such group as I have examined, T may always be considered as the resultant of two non-singular transformations of G_r. Let $T = T_b\,T_a$, in which case

$$TST^{-1} = T_b\,T_a\,S T_a^{-1}\,T_b^{-1} = S'.$$

If, now, T_a is commutative with S,

$$T_b\,T_a\,S T_a^{-1}\,T_b^{-1} = T_b\,S T_b^{-1} = S',$$

and, therefore, S may be continuously transformed into S' by means of the infinitesimal transformation of G_r that generates T_b. In every case examined, it is possible to consider T as the resultant of two non-singular transformations T_b and T_a, T_a being commutative with the given transformation S.

The following examples illustrate the effect of the existence of singular transformations among the transformations of a group G_r upon the interchange, by transformations of G_r, of points on the same invariant manifold relative to G_r. They have been selected from the list given at the end of this paper. The third group considered is the adjoined group of a number of three-parameter projective groups.

Example 1.—Consider the two-parameter projective group of the plane

$$xq, \quad xp + 3yq.$$

The ∞^2 of non-singular transformations are defined by the system of equations

$$x' = e^{c_2} x_3,$$
$$y' = \frac{c_1}{2c_2}(e^{3c_2} - e^{c_1})\,x + e^{3c_2}\,y.$$

The group contains singular transformations which are defined by the system of equations

$$x' = -x,$$
$$y' = Nx - y. \qquad (N \neq 0)$$

Now, a singular transformation T applied to a point p on the line $x = +C$ will transform p to a point p_1 on the line $x = -C$; and, clearly, there is no non-singular transformation T_c among the transformations of the group that has the same effect. If the singular transformation T is applied to a point p on the special invariant $x = 0$, p will be conveyed across the invariant point $x = 0$, $y = 0$. But this can be done by a non-singular transformation whose path-curve is imaginary; for this transformation may be effected by the non-singular transformation

$$x' = -x,$$
$$y' = -y.$$

Therefore, two points p and p_1 in the plane, that lie on opposite sides of and equidistant from the special invariant $x = 0$, cannot *always* be interchanged among themselves continuously by means of the transformations of the group.

Example 2. xq, $xp + q$.

The ∞^2 of non-singular transformations are defined by the equations

$$x' = e^{c_2} x,$$
$$y' = \frac{c_1}{c_2}(e^{c_2} - 1)\,x + y + c_2.$$

The group contains singular transformations which are defined by the system of equations

$$x' = x,$$
$$y' = Nx + y + 2m\pi i. \qquad (N \neq 0,\ m \text{ an integer} \neq 0)$$

A ∞^1 of singular transformations applied to a given point p of general position on the line $x = C$ will convey that point to a given point p_1 of general position on the same line. Nevertheless, there is one non-singular transformation T_c that will do the same, namely,

$$x' = x,$$
$$y' = c_1 x + y.$$

Therefore, two points, p and p_1, of general position in the plane, may always be continuously interchanged by means of transformations of the given group.

Example 3. $\qquad x_3 p_2, \ x_1 p_2, \ x_1 p_1 + 2x_2 p_2.$

The ∞^3 of non-singular transformations are defined by the equations

$$x_1' = e^{c_3} x_1,$$
$$x_2' = \frac{c_2}{c_3}(e^{2c_3} - e^{c_3}) x_1 + e^{2c_3} x_2 + \frac{c_1}{2c_3}(e^{2c_3} - 1) x_3,$$
$$x_3' = x_3,$$

and the ∞^2 of singular transformation are defined by the equations

$$x_1' = -x_1,$$
$$x_2' = Mx_1 + x_2 + Nx_3, \qquad (N \neq 0)$$
$$x_3' = x_3.$$

A ∞^1 of the singular transformations will convey a given point p of general position on the line $x_1 = +C$, $x_3 = k$, to a given point p_1 of general position on the line $x_1 = -C$, $x_3 = k$. Nevertheless, we can find one non-singular transformation that will do the same, namely,

$$x_1' = -x_1,$$
$$x_2' = Ax_1 + x_2,$$
$$x_3' = x_3.$$

For clearly, by a proper choice of A, this transformation T_c has the same effect when applied to a given point as the singular transformation T for any given values of M and N $(N \neq 0)$.

Example 4. $\qquad x_3 p_1, \ x_3 p_2, \ x_1 p_1 + 2x_2 p_2.$

The ∞^3 of non-singular transformations are defined by the equations

$$
\begin{aligned}
x_1' &= e^{c_3} x_1 + \frac{c_1}{c_3}(e^{c_3} - 1)\, x_3, \\
x_2' &= e^{2c_3} x_2 + \frac{c_2}{2c_3}(e^{2c_3} - 1)\, x_3, \\
x_3' &= x_3 .
\end{aligned}
$$

The ∞^2 of singular transformations are defined by the equations

$$
\begin{aligned}
x_1' &= -x_1 + Mx_3, \\
x_2' &= x_2 + Nx_3, \qquad (N \neq 0) \\
x_3' &= x_3 .
\end{aligned}
$$

By means of a singular transformation T a given point p of general position on the plane $x_3 = k$ can be transformed into a given point p_1 of general position in that plane. But it is easily seen that c_1, c_2, and c_3 can be chosen in ∞^1 of ways so that the transformation T_c will produce the same effect.

Example 5. $\quad x_3 p_1,\ x_3 p_2,\ 2x_1 p_2 + 3x_2 p_2 + x_3 p_3 .$

The ∞^3 of non-singular transformations are defined by the equations

$$
\begin{aligned}
x_1' &= e^{2c_3} x_1 + \frac{c_1}{c_3}(e^{2c_3} - e^{c_3})\, x_3, \\
x_2' &= e^{3c_3} x_2 + \frac{c_2}{2c_3}(e^{3c_3} - e^{c_3})\, x_3, \\
x_3' &= e^{c_3} x_3 .
\end{aligned}
$$

The ∞^2 of singular transformations are defined by the equations

$$
\begin{aligned}
x_1' &= x_1 + Mx_3, \\
x_2' &= -x_2 + Nx_3, \qquad (N \neq 0) \\
x_3' &= -x_3 .
\end{aligned}
$$

The singular transformation T, if we regard x_1, x_2, x_3 as Cartesian coordinates, will convey a given point p of general position in the plane $x_3 = +C$ to a point p_1 on the plane $x_3 = -C$; and, clearly, there is no other transformation of the group that will do the same. The points on the special invariant $x_3 = 0$ can be continuously interchanged, for the transformation effected by T can also be effected by the non-singular transformation

$$\begin{aligned} x_1' &= x_1 + Ax_3, \\ x_2' &= -x_2, \\ x_3' &= -x_3. \end{aligned}$$

Therefore, in this group, points on opposite sides of, and equidistant from, the the special invariant plane $x_3 = 0$ cannot *all* be interchanged continuously among themselves.

The following groups enumerated by Lie on pp. 288 and 519 of his "Continuerliche Gruppen" are not properly continuous except in the neighborhood of the identical transformation.*

$$\boxed{q,\ p + xq,\ xp + 2yq} \qquad \boxed{q,\ xq,\ p + yq}$$

$$\boxed{p,\ q,\ xp + (y - x)q} \qquad \boxed{xq,\ xp - yq,\ yp}$$

$$\boxed{p,\ q,\ (a - 1)xp + ayq} \qquad \boxed{q,\ xq,\ xp + ayq}$$

$$\boxed{q,\ yq + p} \qquad \boxed{xq,\ xp + q}$$

$$\boxed{xq,\ xp + ayq\ (a \neq 0,\ 1)}$$

$$\boxed{x_3p_2,\ x_3p_1 + x_1p_2,\ x_2p_2 - x_3p_3 + \beta U} \qquad \boxed{x_3p_1,\ x_3p_2 + U,\ x_1p_1 + \beta U}$$

$$\boxed{x_3p_2,\ x_1p_2,\ x_3p_1 + x_2p_2 + \beta U} \qquad \boxed{x_3p_2,\ x_1p_2 + U,\ x_1p_1 + x_2p_2 + \beta U}$$

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* Cf. Proc. Am. Acad., vol. XXXIII, p. 493.

$$\boxed{x_3p_1,\ x_3p_2,\ x_1p_1+x_1p_2+x_2p_2+\beta U}$$

$$\boxed{x_3p_2,\ x_3p_1+x_2p_2,\ U}$$

$$\boxed{x_1p_2,\ x_1p_1-x_2p_2,\ x_2p_1}$$

$$\boxed{x_1p_2,\ x_1p_1+x_3p_2,\ U}$$

$$\boxed{x_3p_1,\ x_3p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$

$$\boxed{x_3p_2,\ \alpha x_1p_1+\beta x_2p_2,\ U}$$

$$\boxed{x_3p_2;\ x_1p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$

$$\boxed{x_3p_2,\ x_3p_1+x_2p_2+\beta U}$$

$$\boxed{x_3p_2,\ x_1p_1+\alpha U,\ x_2p_2+\beta U}$$

$$\boxed{x_1p_2,\ x_1p_1+x_3p_2+\beta U}$$

$$\boxed{x_3p_2,\ \alpha x_1p_1+\beta x_2p_2+\gamma x_3p_3}$$